

# Casimir Effect for Spherical Shell in de Sitter Space

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February 7, 2008

## Abstract

The Casimir stress on a spherical shell in de Sitter background for massless scalar field satisfying Dirichlet boundary conditions on the shell is calculated. The metric is written in conformally flat form. Although the metric is time dependent, no particles are created. The Casimir stress is calculated for inside and outside of the shell with different backgrounds corresponding to different cosmological constants. The detail dynamics of the bubble depends on different parameter of the model. Specifically, bubbles with true vacuum inside expands if the difference in the vacuum energies is small, otherwise it collapses.

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# 1 Introduction

The Casimir effect is one of the most interesting manifestations of nontrivial properties of the vacuum state in quantum field theory[1,2]. Since its first prediction by Casimir in 1948[3] this effect has been investigated for different fields having different boundary geometries[4-7]. The Casimir effect can be viewed as a polarization of vacuum by boundary conditions or geometry. Therefore, vacuum polarization induced by an gravitational field is also considered as Casimir effect.

A new element which has recently been taking into account is to bring dynamics regarding the boundary conditions or the geometry into this effect. This dynamical Casimir effect have been studied by different authors[8-10]. In the static case perturbation of the quantum state induces vacuum energy and stress but no particle. In contrast, in the vacuum perturbed by dynamical external constraints particles are created. For example, a scalar massless field propagating between two infinite parallel plates moving with constant relative velocity creates particles at the expense of the Casimir energy, or even the motion of a single reflecting boundary can cause such an effect. [8, 9]. Creation of particles by time-dependent gravitational field is another example of such dynamical effects. Taking different possible dynamical effects into account, one may wonder how Casimir effect may correct our view of the early universe. It has been shown, e.g., in[11] that a closed Robertson-Walker space-time in which the only contribution to the stress tensor comes from Casimir energy of a scalar field is excluded. In inflationary models, where the dynamics of bubbles may play a major role, this dynamical Casimir effect has not yet been taken into account.

Casimir effect for spherical shells in the presence of the electromagnetic fields has been calculated several years ago[13,14]. A recent simplifying account of it for the cases of electromagnetic and scalar field with both Dirichlet and Neumann boundary conditions on sphere is given in[14]. The dependence of Casimir energy for scalar and electromagnetic fields with Dirichlet boundary conditions in the presence of a spherical shell is discussed in[12,16]. It has been shown that the Casimir energy in even space dimensions, in contrast to the case of odd dimensions, is divergent. Spinor fields are considered in[17]. Robin's boundary conditions have been studied in[17], where interior and exterior regions are treated separately. It is shown explicitly that although the Casimir energy for interior and exterior of the shell are both divergent irrespective of the number of space dimensions, the total Casimir energy of the shell remains finite for the case of odd space dimensions. Of some interest are cases where the field is confined to the inside of a spherical shell. This is sometimes called as the bag boundary condition. The application of Casimir effect to the bag model is considered for the case of massive scalar field [18] and the Dirac field [19]. The renormalization procedure in the above cases is of interest and we use it for the cases of interest to us.

There are few examples of Casimir effect in curved space-time. Casimir effect for spherical boundary in curved space-time is considered in[21, 22], where the Casimir energy for half of  $s_3$  and  $s_2$  with Dirichlet and Neumann boundary conditions for massless conformal scalar field is calculated analytically using all the existing methods. Casimir effect in the presence of a general relativistic domain wall is considered in [22] and a study on the relation between trace anomaly and the Casimir effect can be found in [23].

None of these cases imply dynamical effects. Our aim is to consider the dynamical Casimir effect on a spherical shell having different vacuums inside and outside representing a bub-

ble in early universe with false/true vacuum inside/outside. Section two is devoted to the Casimir effect for a spherical shell with Dirichlet boundary conditions. In section three we calculate the stress on a spherical shell having constant comoving radius in a de Sitter space. The case of different de Sitter vacuums inside and outside of the shell, using the renormalization method of MIT bag model, is considered in section four. In last section we conclude and summarize the results.

## 2 Scalar Casimir effect for a sphere in flat space-time

Consider the Casimir force due to fluctuations of a free massless scalar field satisfying Dirichlet boundary conditions on a spherical shell in Minkowski space-time [15]. The two-point Green's function  $G(x, t; x', t')$  is defined as the vacuum expectation value of the time-ordered product of two fields

$$G(x, t; x', t') \equiv -i \langle 0 | T \Phi(x, t) \Phi(x', t') | 0 \rangle. \quad (1)$$

It has to satisfy the Dirichlet boundary conditions on the shell:

$$G(x, t; x', t')|_{|x|=a} = 0, \quad (2)$$

where  $a$  is radius of the spherical shell. The stress-energy tensor  $T^{\mu\nu}(x, t)$  is given by

$$T^{\mu\nu}(x, t) \equiv \partial^\mu \Phi(x, t) \partial^\nu \Phi(x, t) - \frac{1}{2} \eta^{\mu\nu} \partial_\lambda \Phi(x, t) \partial^\lambda \Phi(x, t). \quad (3)$$

The radial Casimir force per unit area  $\frac{F}{A}$  on the sphere, called Casimir stress, is obtained from the radial-radial component of the vacuum expectation value of the stress-energy tensor:

$$\frac{F}{A} = \langle 0 | T_{in}^{rr} - T_{out}^{rr} | 0 \rangle|_{r=a}. \quad (4)$$

Taking into account the relation (1) between the vacuum expectation value of the stress-energy tensor  $T^{\mu\nu}(x, t)$  and the Green's function at equal times  $G(x, t; x', t)$  we obtain

$$\frac{F}{A} = \frac{i}{2} \left[ \frac{\partial}{\partial r} \frac{\partial}{\partial r'} G(x, t; x', t)_{in} - \frac{\partial}{\partial r} \frac{\partial}{\partial r'} G(x, t; x', t)_{out} \right]_{x=x', |x|=a}. \quad (5)$$

## 3 Scalar Casimir effect for a sphere in de Sitter space

Consider now a massless scalar field in de Sitter space-time. To make the maximum use of the flat space calculation we use the de Sitter metric in conformally flat form:

$$ds^2 = \frac{\alpha^2}{\eta^2} [d\eta^2 - \sum_{i=1}^3 (dx^i)^2], \quad (6)$$

where  $\eta$ , is the conformal time:

$$- \infty < \eta < 0. \quad (7)$$

Under the conformal transformation in four dimensions the scalar field  $\Phi(x, t)$  is given by

$$\bar{\Phi}(x, \eta) = \Omega^{-1}(x, \eta) \Phi(x, \eta). \quad (8)$$

With the conformal factor given by

$$\Omega(\eta) = \frac{\alpha}{\eta}. \quad (9)$$

And assuming a canonical quantization of the scalar field, and using the creations and annihilations operators  $a_k^\dagger$  and  $a_k$ , the scalar field  $\Phi(x, \eta)$  is then given by

$$\Phi(x, \eta) = \Omega(\eta) \sum_k [a_k \bar{u}_k(\eta, x) + a_k^\dagger \bar{u}_k^*(\eta, x)] \quad (10)$$

The vacuum states associated with the modes  $\bar{u}_k$  defined by  $a_k|\bar{0}\rangle = 0$ , are called conformal vacuum. For the massless scalar field we are considering, the Green's function  $\bar{G}$  associated to the conformal vacuum  $|\bar{0}\rangle$  is given by the flat Feynman Greens function times a conformal factor[8, 25]. Given the flat space Green's function(1), we obtain

$$\bar{G} = -i\langle\bar{0}|T\bar{\Phi}(x, \eta)\bar{\Phi}(x', \eta')|\bar{0}\rangle = \Omega^{-1}(\eta)\Omega^{-1}(\eta')G. \quad (11)$$

Therefore, the stress(5) is given by

$$(\frac{\bar{F}}{A})_{in} = \frac{i}{2}[\frac{\partial}{\partial r}\frac{\partial}{\partial r'}\bar{G}(x, \eta; x', \eta)_{in}]|_{x=x', |x|=a} = \frac{\eta^2}{\alpha^2}(\frac{F}{A})_{in}, \quad (12)$$

and similarly

$$(\frac{\bar{F}}{A})_{out} = \frac{\eta^2}{\alpha^2}(\frac{F}{A})_{out}. \quad (13)$$

Finally, taking the definition (4), we obtain for the total stress on the sphere

$$(\frac{\bar{F}}{A}) = \frac{\eta^2}{\alpha^2}\frac{F}{A}. \quad (14)$$

Now we consider the pure effect of vacuum polarization due to the gravitational field without any boundary conditions. The renormalized stress tensor for massless scalar field in de Sitter space is given by[8, 24]:

$$\langle T_\mu^\nu \rangle = \frac{1}{960\pi^2\alpha^4}\delta_\mu^\nu. \quad (15)$$

The corresponding effective pressure is then

$$P = -\langle T_1^1 \rangle = -\langle T_r^r \rangle = -\frac{1}{960\pi^2\alpha^4}, \quad (16)$$

valid for both in and out side of the sphere. Hence, the effective force on the sphere is zero.

The particle creation in such cases is a delicate problem. The metric (6) has an apparent time dependence. On the other side, it is conformal to Minkowski space and also the scalar field is massless and conformally coupled to de Sitter background. Therefore, we may not expect any particle production. Particle production takes place only when the conformal symmetry is broken by the presence of mass, which provides a length scale for the theory[8]. To see this explicitly, we calculate the corresponding Bogolubov coefficients. Free massless scalar field  $\Phi(x, \eta)$  in Minkowski space-time satisfies the Klein-Gordon equation

$$(\frac{\partial^2}{\partial \eta^2} - \nabla^2)\Phi(x, \eta) = 0. \quad (17)$$

To solve this equation we introduce polar coordinates and seek a solution that has cylindrical symmetry, we seek a solution that is a function only of two variables  $r = |x|$  and  $\theta$ , the angle between  $x$  and  $x'$  so that  $x \cdot x' = rr' \cos \theta$  [15]. In terms of these polar variables (17) becomes

$$\left[ \frac{\partial^2}{\partial \eta^2} - \left( \frac{\partial^2}{\partial r^2} + \frac{2\partial}{r\partial r} + \frac{1}{\sin \theta r^2} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \right) \right] \Phi(r, \theta, \eta) = 0 \quad (18)$$

We can solve (18) using the method of separation of variables. Let

$$\Phi(r, \theta, \eta) = A(r)B(\theta)T(\eta), \quad (19)$$

and

$$T(\eta) = \exp^{-i\omega\eta}. \quad (20)$$

The scalar field  $\Phi(r, \theta, \eta)$  in de Sitter space satisfies

$$(\square + \xi R)\bar{\Phi}(r, \theta, \eta) = 0 \quad (21)$$

where  $\square$  is the Laplace-Beltrami operator for the de Sitter metric, and  $\xi$  is the coupling constant. For conformally coupled field in four dimension  $\xi = \frac{1}{6}$ , and  $R$ , the Ricci scalar curvature, is given by

$$R = 12\alpha^{-2}. \quad (22)$$

Now, the Bogolubov transformation, given by

$$u_k^{in}(r, \theta, \eta) = \alpha_k u_k^{out}(r, \theta, \eta) + \beta_k u_{-k}^{*out}(r, \theta, \eta), \quad (23)$$

defines the Bogolubov coefficients  $\alpha_k$  and  $\beta_k$ . Here "in" and "out" corresponds to  $(\eta \rightarrow -\infty)$  and  $(\eta \rightarrow T < 0)$  respectively. Taking into account the separation of variables (19) we obtain from (23):

$$T_k^{in}(\eta) = \alpha_k T_k^{out}(\eta) + \beta_k T_k^{*out}(\eta). \quad (24)$$

Due to (8) and (20) we may write

$$\eta \exp(-i\omega\eta) = \alpha_k \eta \exp(-i\omega\eta) + \beta_k \eta \exp(i\omega\eta). \quad (25)$$

Therefore

$$\alpha_k = 1 \quad \beta_k = 0. \quad (26)$$

But the expectation value of the number operator  $N_k = a_k^\dagger a_k$  for the number of  $\bar{u}_k$ -mode particles in the state  $|\bar{0}\rangle$  is given by [8]

$$\langle \bar{0} | N_k | \bar{0} \rangle = \sum_j |\beta_{jk}|^2, \quad (27)$$

which is zero in our case. Therefore there is no particle production in our case.

## 4 Spherical shell with different vacuum inside and outside

Now, assume there are different vacuums in- and out-side, corresponding to  $\alpha_{in}$  and  $\alpha_{out}$  for the metric (6). It is then more suitable to use the following relation for the stress on the shell [13]:

$$\frac{F}{A} = \langle T_{rr} \rangle_{in} - \langle T_{rr} \rangle_{out} = \frac{-1}{4\pi a^2} \frac{\partial E}{\partial a}, \quad (28)$$

where  $E$  is Casimir energy due to boundary conditions. The Casimir energy  $E$  is the sum of Casimir energies  $E_{in}$  and  $E_{out}$  for inside and outside of the shell. As described in introduction, Casimir energies in-side and out-side of the shell are divergent individually. In flat space when we calculate the total Casimir energy, we add interior and exterior energies to each other. Now divergent parts will cancel each other out, when interior and exterior background are the same, like the case mentioned in last section, we get the above result again.

In flat space-time for massless scalar field with Dirichlet boundary conditions the Casimir energy in- and out-side of a spherical shell is given by[17]

$$E_{in} = \frac{1}{2a} \left( 0.008873 + \frac{0.001010}{\epsilon} \right) \quad E_{out} = \frac{-1}{2a} \left( 0.003234 + \frac{0.001010}{\epsilon} \right). \quad (29)$$

Each of the energies for in-and out-side of the shell is divergent, and cutoff dependent. But the Casimir energy, which is the sum of  $E_{in}$  and  $E_{out}$  is independent of the cutoff  $\epsilon$  and finite. In the case of different in- and out-side backgrounds, the boundary part of the total Casimir energy is calculated to be

$$\bar{E}_{in} = \frac{\eta^2}{2a\alpha_{in}^2} \left( c_1 + \frac{c_{1'}}{\epsilon} \right) \quad \bar{E}_{out} = \frac{\eta^2}{2a\alpha_{out}^2} \left( c_2 - \frac{c_{1'}}{\epsilon} \right), \quad (30)$$

where,  $c_1 = 0.008873$ ,  $c_2 = -0.003234$ ,  $c_{1'} = 0.001010$ . In this case, we have

$$\bar{E} = \bar{E}_{in} + \bar{E}_{out} = \frac{\eta^2}{2a} \left( \frac{c_1}{\alpha_{in}^2} + \frac{c_2}{\alpha_{out}^2} \right) + \frac{\eta^2 c_{1'}}{2a\epsilon} \left( \frac{1}{\alpha_{in}^2} - \frac{1}{\alpha_{out}^2} \right). \quad (31)$$

Therefore the Casimir energy for this general case becomes cutoff dependent and divergent. To renormalize the Casimir energy  $\bar{E}$ , we use a procedure similar to that of the bag model[18-19]. The classical energy of a spherical shell, or bubble, immersed in a cosmological background, as we are considering, maybe written as

$$E_{(class)} = pa^3 + \sigma a^2 + Fa + K + \frac{h}{a}, \quad (32)$$

where the meaning of the terms proportional to  $P$  and  $\sigma$  is obvious. The third and forth terms on the right hand side of the above equation corresponds to the curvature and cosmological term respectively. The last term is considered as non-vanishing because of the intuition obtained from the calculation of the Casimir effect in the last section. There we have seen that the Casimir energy on each side of the bubble is proportional to  $\frac{1}{a}$ . Terms proportional to other powers of  $a$  is therefore not expected. Now, in our case the total Casimir energy is divergent and therefore we have to renormalize the parameter  $h$ . The total energy of the shell maybe written as

$$\tilde{E}_{in} = \bar{E}_{in} + E_{(class)} \quad \tilde{E}_{out} = \bar{E}_{out} + E_{(class)}. \quad (33)$$

The renormalization can be achieved now by shifting the parameter  $h$  of  $E_{(class)}$  by an amount which cancels the divergent contribution. For inside and outside we have

$$h \rightarrow h - \frac{\eta^2 c_{1'}}{2a\epsilon\alpha_{in}^2} \quad h \rightarrow h + \frac{\eta^2 c_{1'}}{2a\epsilon\alpha_{out}^2}. \quad (34)$$

We finally obtain for the total zero point energy of our system:

$$\bar{E} = \frac{\eta^2}{2a} \left( \frac{c_1}{\alpha_{in}^2} + \frac{c_2}{\alpha_{out}^2} \right). \quad (35)$$

In contrast to the Minkowski- and de Sitter-space the Casimir energy can now be positive or negative depending on the value of  $\alpha$  in- and out-side of the bubble. The stress on the shell due to boundary conditions is then obtained(28):

$$\frac{\bar{F}}{A} = \frac{-1}{4\pi a^2} \frac{\partial \bar{E}}{\partial a} = \frac{\eta^2}{8\pi a^4} \left( \frac{c_1}{\alpha_{in}^2} + \frac{c_2}{\alpha_{out}^2} \right). \quad (36)$$

As expected, one obtain the previous result (14) for the case  $\alpha_{in} = \alpha_{out}$ . Now, the effective pressure created by gravitational part(16), is different for different part of space-time:

$$P_{in} = - \langle T_r^r \rangle_{in} = \frac{-1}{960\pi^2 \alpha_{in}^4} \quad P_{out} = - \langle T_r^r \rangle_{out} = \frac{-1}{960\pi^2 \alpha_{out}^4} \quad (37)$$

Therefore, the gravitational pressure over shell,  $P_G$ , is given by

$$P_G = P_{in} - P_{out} = \frac{-1}{960\pi^2} \left( \frac{1}{\alpha_{in}^4} - \frac{1}{\alpha_{out}^4} \right). \quad (38)$$

Call the stress due to the boundary  $P_B$ . The total pressure on the shell,  $P$ , is then given by

$$P = P_G + P_B = \frac{-1}{960\pi^2} \left( \frac{1}{\alpha_{in}^4} - \frac{1}{\alpha_{out}^4} \right) + \frac{\eta^2}{8\pi a^4} \left( \frac{c_1}{\alpha_{in}^2} + \frac{c_2}{\alpha_{out}^2} \right). \quad (39)$$

Noting the relation  $\alpha^2 = \frac{3}{\Lambda}$ , we may write the total pressure in terms of the cosmological constants:

$$P = \frac{-1}{2880\pi^2} (\Lambda_{in}^2 - \Lambda_{out}^2) + \frac{\eta^2}{24\pi a^4} (c_1 \Lambda_{in} + c_2 \Lambda_{out}). \quad (40)$$

This total pressure may be both negative or positive. Note that this is just the pressure due to quantum effects. Therefore the following discussions should be taken cautiously. To see the different possible cases, let us first assume

$$c_1 \Lambda_{in} + c_2 \Lambda_{out} > 0, \quad (41)$$

then  $P_B > 0$ , i.e. the Casimir force on the bubble is repulsive. Given a false vacuum inside and true vacuum outside, i.e.  $\Lambda_{in} > \Lambda_{out}$ , then the gravitational part is negative. Therefore the total pressure may be either negative or positive. Given  $P > 0$  initially, then the initial expansion of the bubble leads to a change of the Casimir part of the pressure. This change, depending on the detail of the dynamics of the bubble, may be an increase or a decrease. Therefore, the initial expansion of the bubble may end and a phase of contraction could begin. For  $P < 0$ , there is an initial contraction which ends up at a minimum radius. For the case of true vacuum inside and false vacuum outside, i.e.  $\Lambda_{in} < \Lambda_{out}$ , which is more interesting cosmologically, the total pressure is always positive. Therefore the bubble expands without any limit.

Now consider the case

$$c_1 \Lambda_{in} + c_2 \Lambda_{out} < 0. \quad (42)$$

Noting that  $|c_1| > |c_2| = -c_2$ , it is seen that the inside has to be a true vacuum, i.e.  $\Lambda_{in} < \Lambda_{out}$ . Therefore, the total pressure may be either negative or positive. For  $P > 0$ , the initial expansion of the bubble may be stopped or not depending on the detail of the dynamics. For  $P < 0$ , the bubble contracts and the total pressure remains negative. Hence, it ends up to a collapse of the bubble.

## 5 Conclusion

Spherical bubbles with different vacuums in- and out-side, corresponding to different de Sitter space-times, are encountered in inflationary scenarios. To study the dynamics of such bubbles one should know the Casimir effect on them. We have considered a spherical shell in de Sitter background with a massless scalar field, coupled conformally to the background, satisfying the Dirichlet boundary conditions. Although the metric is time dependent we could show that for a bubble with constant comoving radius no particle is created. Our calculation shows that the detail dynamics of the bubble depends on different parameters and all cases of contraction, expansion and collapse may appear. The interesting case of true vacuum inside leads to an expansion if the difference of two vacuum energies is small. Otherwise the bubble contracts and it leads to the collapse of the bubble.

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